

Analytic fields on compact balanced Hermitian manifolds

G. GANCHEV ^{*} S.IVANOV [†]

Abstract

On a Hermitian manifold we construct a symmetric $(1,1)$ - tensor H using the torsion and the curvature of the Chern connection. On a compact balanced Hermitian manifold we find necessary and sufficient conditions in terms of the tensor H for a harmonic 1-form to be analytic and for an analytic 1- form to be harmonic. We prove that if H is positive definite then the first Betti number $b_1 = 0$ and the Hodge number $h^{1,0} = 0$. We obtain an obstruction to the existence of Killing vector fields in terms of the Ricci tensor of the Chern connection. We prove that if the Chern form of the Chern connection on a compact balanced Hermitian manifold is non-positive definite then every Killing vector field is analytic; if moreover the Chern form is negative definite then there are no Killing vector fields.

Running title: Analytic fields on balanced manifolds

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1 Introduction

On a compact Riemannian manifold (\mathbf{M}, g) certain objects of geometric interest, such as Killing vector fields and harmonic 1-forms must satisfy additional differential equations when appropriate Levi-Civita curvature conditions are imposed. This leads to obstructions to the existence of these objects, known as Bochner-type vanishing theorems. For example, the well known theorems of Bochner state that if the Ricci tensor is non-negative (resp. non-positive) then every harmonic 1-form (resp. every Killing vector field) must be parallel and if the Ricci tensor is negative (resp. positive) at one point then there are no harmonic 1-forms (resp. no Killing vector fields). By the Hodge theory the nonexistence of harmonic 1-forms leads to the vanishing of the first de Rham cohomology group $H^1(\mathbf{M}, \mathbf{R})$.

On a Hermitian manifold (\mathbf{M}, g, J) another objects of geometric interest are holomorphic vector fields and holomorphic forms. Closely related to the hermitian structure is the Chern connection. On a compact Hermitian manifold holomorphic vector fields and holomorphic $(p, 0)$ -forms have to satisfy additional differential equations when appropriate curvature conditions on the Chern connection are imposed. This also leads to some obstructions to the existence of these objects, so called vanishing theorems for holomorphic sections. For example, if the mean curvature of the Chern connection (in the sense of [13]) on a compact Hermitian manifold is non-negative (resp. non-positive) then every holomorphic $(1, 0)$ -form (resp. every holomorphic vector field) is parallel with respect to the Chern connection and if the mean curvature is positive (resp. negative) at one point then there are no holomorphic $(1, 0)$ -forms (resp. no holomorphic vector fields) (for more general formulations see [14, 8]). By the Hodge theory (see [11]) this leads to the vanishing of the Dolbeault cohomology group $H^{1,0}(\mathbf{M}, \mathbf{C})$.

In the Kaehler geometry there is a close relation between the metric objects (Killing vector fields, harmonic 1-forms) on one hand and the holomorphic objects (holomorphic vector fields, holomorphic $(1, 0)$ -forms) on the other hand. For a Kaehler manifold the Levi-Civita connection coincides with the Chern connection. Then the mean curvature of the Chern connection is exactly the Ricci tensor. Thus, on a compact Kaehler manifold the positive (resp. negative) definiteness of the Ricci tensor is an obstruction to the existence not only of harmonic 1-forms (resp. Killing vector fields) but also to the existence of holomorphic $(1, 0)$ -forms (resp. holomorphic vector fields). Moreover, on a compact Kaehler manifold an 1-form is harmonic iff it is analytic (i.e its $(1, 0)$ -part is holomorphic) and every Killing vector field is analytic (see [19]). So the first Betti number $b_1 = \dim H^1(\mathbf{M}, \mathbf{R})$ vanishes iff the Hodge number $h^{1,0} = \dim H^{1,0}(\mathbf{M}, \mathbf{C})$ vanishes.

In general, on a compact Hermitian manifold there is no remarkable relation between Killing and holomorphic vector fields and between harmonic 1-forms and holomorphic $(1, 0)$ -forms.

In this paper we consider compact balanced Hermitian manifolds and try to find

a connection between the metric objects and the holomorphic objects mentioned above. Balanced manifolds have been studied intensively in [15, 1, 2, 3]; in [8] they are called semi-Kaehler of special type. This class of manifolds includes the class of Kaehler manifolds but also many important classes of non-Kaehler manifolds, such as: complex solvmanifolds, twistor spaces of oriented riemannian 4-manifolds, 1-dimensional families of Kaehler manifolds (see [15]), hermitian compact manifolds with flat Chern connection (see [8]), twistor spaces of oriented distinguished Weyl structure on compact self-dual 4-manifolds [9], twistor spaces of quaternionic Kaehler manifolds [16, 4], manifolds obtained as modification of compact Kaehler manifolds [1] and of compact balanced manifolds [2], see also [3].

We construct on a Hermitian manifold a symmetric $(1,1)$ - tensor H using the torsion and the curvature of the Chern connection. On a compact balanced Hermitian manifold we give in Theorem 4.2 necessary and sufficient conditions in terms of the tensor H for a harmonic 1-form to be analytic and for an analytic 1-form to be harmonic. This allows us to obtain a vanishing theorem of Bochner type on compact balanced Hermitian manifolds (Theorem 4.7). We prove that if H is positive definite then $b_1 = 0$ and $h^{1,0} = 0$. We obtain an obstruction to the existence of Killing vector fields in terms of the Ricci tensor (or the Chern form) of the Chern connection. In Theorem 3.8 we prove that if the Chern form of the Chern connection on a compact balanced Hermitian manifold is non-positive then every Killing vector field is analytic; if moreover the Chern form is negative then there are no Killing vector fields.

It is well known that on a compact Riemannian manifold a smooth vector field is Killing iff it is affine with respect to the Levi-Civita connection [19]. Thus, on a compact Kaehler manifold every affine vector field is analytic. On a compact balanced Hermitian manifold we find necessary and sufficient conditions in terms of the Lie derivative of the Chern connection for a smooth vector field to be analytic. In particular we prove that every affine vector field with respect to the Chern connection is analytic on a compact balanced Hermitian manifold.

2 Preliminaries

Let (\mathbf{M}, J, g) be a $2n$ -dimensional Hermitian manifold with complex structure J and Riemannian metric g . The algebra of all C^∞ vector fields on \mathbf{M} will be denoted by \mathbf{XM} . The complex structure J on the tangent bundle \mathbf{TM} of \mathbf{M} induces a splitting of the complexified tangent bundle $\mathbf{T}_c\mathbf{M}$ into two complementary subbundles, conjugate to each other: $\mathbf{T}_c\mathbf{M} = \mathbf{T}^{(1,0)}\mathbf{M} + \mathbf{T}^{(0,1)}\mathbf{M}$. The elements of $\mathbf{T}^{(1,0)}\mathbf{M}$ (resp. $\mathbf{T}^{(0,1)}\mathbf{M}$) are the (complex) tangent vectors of type $(1,0)$ (resp. of type $(0,1)$). Each real tangent vector field X can be expressed in a unique way as a sum: $X = U + \bar{U}$, where $U = \frac{1}{2}(X - \sqrt{-1}JX) \in \mathbf{T}^{(1,0)}\mathbf{M}$ and $\bar{U} = \frac{1}{2}(X + \sqrt{-1}JX) \in \mathbf{T}^{(0,1)}\mathbf{M}$. With respect to local holomorphic coordinates $\{z^\alpha\}$, ($\alpha = 1, \dots, n$) we

have $U = X^\alpha \frac{\partial}{\partial z^\alpha}, \bar{U} = X^{\bar{\alpha}} \frac{\partial}{\partial \bar{z}^\alpha}$ (summation convention is assumed further in the paper). The induced complex structure on the cotangent bundle $\mathbf{T}^*\mathbf{M}$ (also denoted by J) is defined by: $(J\omega)(X) = -\omega(JX)$, where ω is a real 1-form and X is a real vector field on \mathbf{M} . For the complexified cotangent bundle $\mathbf{T}_c^*\mathbf{M}$ we have the splitting: $\mathbf{T}_c^*\mathbf{M} = \Lambda^{1,0}(\mathbf{M}) + \Lambda^{0,1}(\mathbf{M})$. The elements of $\Lambda^{1,0}(\mathbf{M})$ (resp. $\Lambda^{0,1}(\mathbf{M})$) are the (complex) 1-forms of type $(1,0)$ (resp. of type $(0,1)$). Each real 1-form ω can be expressed in a unique way as a sum $\omega = \beta + \bar{\beta}$, where $\beta = \frac{1}{2}(\omega - \sqrt{-1}J\omega) \in \Lambda^{1,0}(\mathbf{M})$; $\bar{\beta} = \frac{1}{2}(\omega + \sqrt{-1}J\omega) \in \Lambda^{0,1}(\mathbf{M})$. With respect to local holomorphic coordinates we have $\beta = \omega_\alpha dz^\alpha, \bar{\beta} = \omega_{\bar{\alpha}} d\bar{z}^{\bar{\alpha}}$. In the whole paper all tensors and connections will be extended complex multilinearly to the complexification $\mathbf{T}_c\mathbf{M}$ of $\mathbf{T}\mathbf{M}$.

The Kaehler form Ω on \mathbf{M} is defined by $\Omega(X, Y) = g(JX, Y)$; $X, Y \in \mathbf{X}\mathbf{M}$. The Lee form θ of the Hermitian structure is defined by $\theta = -\delta\Omega \circ J$.

The Levi-Civita connection and the canonical Chern connection (the Hermitian connection) will be denoted by ∇ and D , respectively. We recall that the Chern connection D is the unique linear connection preserving the metric and the complex structure with torsion tensor T , having the following property: $T(JX, Y) = T(X, JY)$. This implies (e.g. [5]):

$$(2.1) \quad T(JX, Y) = JT(X, Y), \quad X, Y \in \mathbf{X}\mathbf{M}.$$

The two connections are related by the following identity

$$(2.2) \quad g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2}d\Omega(JX, Y, Z), \quad X, Y, Z \in \mathbf{X}\mathbf{M}.$$

Let e_1, \dots, e_{2n} be an orthonormal local basis on \mathbf{M} . We consider the following Ricci-type tensors associated with the curvature tensor K of the Chern connection:

$$k(X, Y) = -\frac{1}{2} \sum_{j=1}^{2n} g(K(X, JY)e_j, Je_j); \quad k^*(X, Y) = -\frac{1}{2} \sum_{j=1}^{2n} g(K(e_j, Je_j)X, JY);$$

$$s(X, Y) = \sum_{j=1}^{2n} g(K(e_j, X)Y, e_j).$$

The $(1,1)$ -form κ corresponding to the tensor k represents the first Chern class of \mathbf{M} (further we will call it the Chern form) and the $(1,1)$ -form κ^* corresponding to the tensor k^* is the mean curvature of the holomorphic tangent bundle $\mathbf{T}^{(1,0)}\mathbf{M}$ with the hermitian metric g .

Using the torsion tensor T of the Chern connection we construct another remarkable symmetric $(1,1)$ tensor as follows

$$t(X, Y) = \sum_{\alpha, \beta=1}^n g(T(E_\alpha, E_\beta), X)g(T(E_{\bar{\alpha}}, E_{\bar{\beta}}), Y),$$

where $E_1, \dots, E_n, E_{\bar{1}}, \dots, E_{\bar{n}}$ is a hermitian basis on $\mathbf{T}_c\mathbf{M}$. From the definition and (2.1) it follows that t is symmetric, J -invariant positive semi-definite tensor i.e. $t(X, Y) = t(Y, X) = t(JX, JY)$; $t(X, X) \geq 0, X, Y \in \mathbf{XM}$

As we shall see below, the tensor t plays an important role for the coincidence of harmonic 1-forms with analytic 1-forms.

For a real 1-form ω we denote by $\omega^\#$ the corresponding vector field defined by: $\omega(Y) := g(\omega^\#, Y), Y \in \mathbf{XM}$ and for a real vector field X we denote by ω_X the corresponding real 1-form defined by: $\omega_X(Y) := g(X, Y)$. If $\omega = \omega_\alpha dz^\alpha$ is an $(1, 0)$ -form (resp. $\omega_{\bar{\alpha}} dz^{\bar{\alpha}}$ is a $(0, 1)$ -form) then $\omega^\# = g^{\alpha\bar{\beta}} \omega_\alpha \frac{\partial}{\partial z^\beta}$ is the corresponding $(0, 1)$ -vector field (resp. $\omega^\# = g^{\alpha\bar{\beta}} \omega_{\bar{\beta}} \frac{\partial}{\partial z^\alpha}$ is the $(1, 0)$ -vector field) and if $X = X^\alpha \frac{\partial}{\partial z^\alpha}$ is an $(1, 0)$ -vector field (resp. $X = X^{\bar{\alpha}} \frac{\partial}{\partial z^{\bar{\alpha}}}$ is a $(0, 1)$ -vector field) then $\omega_X = g_{\alpha\bar{\beta}} X^\alpha dz^{\bar{\beta}}$ is a $(0, 1)$ -form (resp. $\omega_X = g_{\bar{\alpha}\beta} X^{\bar{\alpha}} dz^\beta$ is an $(1, 0)$ -form).

For a real 1-form ω using (2.2) we calculate

$$(2.3) \quad (\nabla_X \omega)Y = (D_X \omega)Y - \frac{1}{2} d\Omega(JX, Y, \omega^\#), \quad X, Y \in \mathbf{XM}.$$

From (2.3) it follows that

$$(2.4) \quad \delta\omega = - \sum_{i=1}^{2n} (D_{e_i} \omega) e_i - \theta(\omega^\#).$$

We recall here the definition of a balanced manifold and some characterizations given in [15] and [8], for completeness:

DEFINITION: A *balanced* manifold \mathbf{M} is a compact complex n -manifold which satisfies one of the following equivalent conditions:

- i) \mathbf{M} admits a hermitian structure (g, J) such that $d\Omega^{n-1} = 0$;
- ii) \mathbf{M} admits a hermitian structure (g, J) such that $\delta\Omega = 0$;
- iii) \mathbf{M} admits a hermitian structure (g, J) such that $\theta = 0$;
- iv) \mathbf{M} admits a hermitian structure (g, J) such that $\Delta_\partial f = \Delta_{\bar{\partial}} f = \frac{1}{2} \Delta_d f$, for any smooth function f on \mathbf{M} , where $\Delta_\partial, \Delta_{\bar{\partial}}, \Delta_d f$ denote the Laplacians with respect to the operators $\partial, \bar{\partial}, d$, respectively.
- v) there are no non-zero positive $(n-1, n-1)$ -currents on \mathbf{M} which are $(n-1, n-1)$ -components of boundaries.

In this paper we shall use essentially iii) and iv).

On balanced manifolds the first Ricci tensor k coincides with the third Ricci tensor s (see e.g. [6]).

If (\mathbf{M}, J, g) is a balanced manifold then the equality (2.4) takes the form

$$(2.5) \quad \delta\omega = - \sum_{i=1}^{2n} (D_{e_i} \omega) e_i.$$

3 Analytic and Killing vector fields

A real vector field ξ is said to be *analytic* if $L_\xi J = 0$, where L_ξ denotes the Lie derivative with respect to ξ . The vector field ξ is analytic iff the $(2, 0)$ part of $D\omega_\xi$ vanishes. In local holomorphic coordinates this condition can be written as follows

$$(3.6) \quad D_\alpha \xi_\beta = 0,$$

i.e. the $(1, 0)$ -part of ξ is a holomorphic vector field. We consider the following real 1-form ω defined by

$$\omega = \xi^\beta D_\alpha \xi_\beta dz^\alpha + \xi^{\bar{\beta}} D_{\bar{\alpha}} \xi_{\bar{\beta}} d\bar{z}^{\bar{\alpha}}.$$

Using essentially that $DJ = 0$ and (2.3), we find

$$\delta\omega = -\|D_\alpha \xi_\beta\|^2 - 2\operatorname{Re} \left[\xi^\beta D^\alpha D_\alpha \xi_\beta + \xi^\beta \theta^\alpha D_\alpha \xi_\beta \right],$$

where $2\operatorname{Re}(f)$ denotes the real part of a complex valued function f and $\|D_\alpha \xi_\beta\|^2$ is the norm of the $(2, 0) + (0, 2)$ -part of $D\omega_\xi$. The norm of the $(1, 1)$ -part of $D\omega_\xi$ will be denoted by $\|D_\alpha \xi_{\bar{\beta}}\|^2$.

Thus, on a compact Hermitian manifold we have the formula

$$(3.7) \quad \int_{\mathbf{M}} \left\{ \|D_\alpha \xi_\beta\|^2 + 2\operatorname{Re} \left[\xi^\beta D^\alpha D_\alpha \xi_\beta + \xi^\beta \theta^\alpha D_\alpha \xi_\beta \right] \right\} dV = 0.$$

Taking into account the Ricci formula

$$(3.8) \quad D^\alpha D_\alpha \xi_\beta = D_\alpha D^\alpha \xi_\beta + k_{\beta\bar{\sigma}}^* \xi^{\bar{\sigma}}$$

from (3.7) we obtain

Proposition 3.1 *Let ξ be a real vector field on a compact Hermitian manifold. The following conditions are equivalent:*

- i) ξ is analytic;
- ii) $D^\alpha D_\alpha \xi_\beta + \theta^\alpha D_\alpha \xi_\beta = 0$;
- iii) $D_\alpha D^\alpha \xi_\beta + k_{\beta\bar{\sigma}}^* \xi^{\bar{\sigma}} + \theta^\alpha D_\alpha \xi_\beta = 0$.

Hence, on a compact balanced Hermitian manifold we have [5]

Proposition 3.2 *Let ξ be a real vector field on a compact balanced Hermitian manifold. The following conditions are equivalent:*

- i) ξ is analytic;
- ii) $D^\alpha D_\alpha \xi_\beta = 0$;
- iii) $D_\alpha D^\alpha \xi_\beta + k_{\beta\bar{\sigma}}^* \xi^{\bar{\sigma}} = 0$.

On every compact balanced Hermitian manifold the Ricci formula (3.8) leads to the following integral formula

$$(3.9) \quad \int_{\mathbf{M}} \|D_{\alpha}\xi_{\beta}\|^2 dV = \int_{\mathbf{M}} \|D_{\alpha}\xi_{\bar{\beta}}\|^2 dV - \int_{\mathbf{M}} k^*(\omega^{\#}, \omega^{\#}) dV.$$

DEFINITION. A real vector field ξ on a Hermitian manifold is said to be *affine Hermitian* if it is affine vector field with respect to the Chern connection D , i.e $L_{\xi}D = 0$. If the linear connection $L_{\xi}D$ preserves the complex structure J , i.e.

$$(3.10) \quad (L_{\xi}D) \circ J = J \circ (L_{\xi}D),$$

then we call ξ a *complex Hermitian* vector field.

In local holomorphic coordinates the condition (3.10) is equivalent to the equations

$$(L_{\xi}D)_{\alpha\bar{\beta}}^{\lambda} = 0; \quad (L_{\xi}D)_{\alpha\beta}^{\bar{\lambda}} = 0.$$

Using the general formulas expressing $L_{\xi}D$ with the torsion and curvature (see [18]) we find

$$(L_{\xi}D)_{\alpha\bar{\beta}}^{\lambda} = D_{\alpha}D_{\bar{\beta}}\xi^{\lambda}; \quad (L_{\xi}D)_{\alpha\beta}^{\bar{\lambda}} = D_{\alpha}D_{\beta}\xi^{\bar{\lambda}}.$$

Applying Proposition 3.2 we obtain

Theorem 3.3 *Let ξ be a real vector field on a compact balanced Hermitian manifold. The following conditions are equivalent:*

- i) ξ is analytic;*
- ii) ξ is complex Hermitian.*

In the Kaehler case the Chern connection coincides with the Levi-Civita connection . From Theorem 3.3 we have

Corollary 3.4 *A real vector field ξ on a compact Kaehler manifold is analytic iff $L_{\xi}\nabla$ preserves the complex structure.*

Every affine Hermitian vector field is complex Hermitian. From Theorem 3.3 we obtain

Theorem 3.5 *Every affine Hermitian vector field on a compact balanced Hermitian manifold is analytic.*

This result extends the well known result that every affine vector field on a compact Kaehler manifold is Killing and hence, it is analytic.

We recall that a real vector field ξ is said to be a Killing vector field if $L_{\xi}g = 0$. This condition is equivalent to

$$(3.11) \quad (\nabla_X\omega_{\xi})Y + (\nabla_Y\omega_{\xi})X = 0, X, Y \in \mathbf{XM}.$$

The condition (3.11) implies $\delta\omega_\xi = 0$.

Let ξ be a real vector field. We consider the following real 1-form

$$\phi = \xi^\beta D_\beta \xi_\alpha dz^\alpha + \xi^{\bar{\beta}} D_{\bar{\beta}} \xi_{\bar{\alpha}} dz^{\bar{\alpha}}.$$

Using (2.4) we find

$$(3.12) \quad \delta\phi = -2Re \left[D_\beta \xi_\alpha D^\alpha \xi^\beta \right] - 2Re \left[\xi^\beta D^\alpha D_\beta \xi_\alpha + \xi^\beta \theta^\alpha D_\beta \xi_\alpha \right].$$

Using the Ricci identities and (2.4) we calculate

$$(3.13) \quad 2Re \left[\xi^\beta D^\alpha D_\beta \xi_\alpha \right] = s(\xi, \xi) - \frac{1}{2} \xi \delta\omega_\xi - \frac{1}{2} J\xi \delta\omega_{J\xi} - \frac{1}{2} \xi \theta(\xi).$$

On a compact Hermitian manifold we derive from (3.13) and (3.12) the following formula

$$\begin{aligned} \int_{\mathbf{M}} \left\{ 2Re \left[D_\beta \xi_\alpha D^\alpha \xi^\beta \right] + s(\xi, \xi) - \frac{1}{2} (\delta\omega_\xi)^2 - \frac{1}{2} (\delta\omega_{J\xi})^2 \right\} dV - \\ - \int_{\mathbf{M}} \left\{ \frac{1}{2} \xi \theta(\xi) + 2Re \left[\xi^\beta \theta^\alpha D_\beta \xi_\alpha \right] \right\} dV = 0. \end{aligned}$$

Thus, we proved

Proposition 3.6 *If ξ is a real vector field on a compact balanced Hermitian manifold, then*

$$(3.14) \quad \int_{\mathbf{M}} \left\{ 2Re \left[D_\beta \xi_\alpha D^\alpha \xi^\beta \right] + k(\xi, \xi) - \frac{1}{2} (\delta\omega_\xi)^2 - \frac{1}{2} (\delta\omega_{J\xi})^2 \right\} dV = 0.$$

Now, let ξ be a Killing vector field. Using (2.3) from (3.11) we get

$$(3.15) \quad D_\alpha \xi_\beta + D_\beta \xi_\alpha = 0.$$

Using (3.15) from Proposition 3.6 we get

Proposition 3.7 *If ξ is a Killing vector field on a compact balanced Hermitian manifold, then*

$$(3.16) \quad \int_{\mathbf{M}} \left\{ \|D_\beta \xi_\alpha\|^2 - k(\xi, \xi) + \frac{1}{2} (\delta\omega_{J\xi})^2 \right\} dV = 0.$$

As a corollary we obtain the following theorem of Bochner type

Theorem 3.8 *Let (\mathbf{M}, g, J) be a compact balanced Hermitian manifold.*

i) If the Chern form κ is non-positive definite, then every Killing vector field ξ on \mathbf{M} is analytic and satisfies the equality

$$k(\xi, \xi) = \delta(\omega_{J\xi}) = 0.$$

ii) If the Chern form κ is negative definite, then there are no Killing vector fields other than zero, i.e. the group of isometries of (\mathbf{M}, g, J) is discrete.

Remark 3.9 *If the Chern form κ is negative definite then the first Chern class is negative. By the theorem of S.Kobayashi [12] it follows that there are no holomorphic vector fields and hence there are no Killing vector fields with respect to any Kaehler metric on \mathbf{M} .*

EXAMPLE. Let $\mathbf{M} = G/\Gamma$ be a compact quotient of a complex Lie group G with respect to its discrete subgroup Γ . It is well known that every compact Hermitian manifold with flat Chern connection is isomorphic with \mathbf{M} endowed with its flat hermitian structure [10]. If the group G is non-abelian, then \mathbf{M} is a non-Kaehler balanced Hermitian manifold. Since $k = k^* = 0$, then every Killing vector field is analytic and by (3.9) it is parallel. (There is a general result of P.Gauduchon [8] that on a compact Hermitian manifold with negative semi-definite mean curvature k^* every analytic vector field is parallel (see also [13])).

Remark 3.10 *It is clear from above that Proposition 3.7 and Theorem 3.8 are valid if we only assume the condition (3.15) which is weaker than the Killing condition (3.11)*

4 Harmonic and analytic 1-forms

A real 1-form ω is *analytic* if its $(1,0)$ -part is holomorphic. In terms of the Chern connection this condition is equivalent to the condition

$$(4.17) \quad D_\alpha \omega_{\bar{\beta}} = 0$$

We recall that a real 1-form ω on a Riemannian manifold is *harmonic* if it is closed and co-closed, i.e.

$$(4.18) \quad d\omega = 0; \quad \delta\omega = 0.$$

Using (2.3) the condition $d\omega = 0$ is equivalent to the following two conditions:

$$(4.19) \quad D_\alpha \omega_{\bar{\beta}} - D_{\bar{\beta}} \omega_\alpha = 0.$$

$$(4.20) \quad D_\alpha \omega_\beta - D_\beta \omega_\alpha = -T_{\alpha\beta}^\sigma \omega_\sigma.$$

The condition (4.19) implies

$$(4.21) \quad \delta(J\omega) = 0.$$

We are going to obtain necessary and sufficient conditions for a harmonic 1-form to be analytic and for a holomorphic 1-form to be harmonic on a compact balanced Hermitian manifold. For this purpose we need some integral formulas.

Proposition 4.1 *For every real 1-form ω on a compact balanced Hermitian manifold the following integral formulas are valid:*

$$(4.22) \quad \int_{\mathbf{M}} \frac{1}{2} \|D_\alpha \omega_\beta - D_\beta \omega_\alpha + T_{\alpha\beta}^\sigma \omega_\sigma\|^2 dV =$$

$$\begin{aligned}
& \int_{\mathbf{M}} \left[\|D_\alpha \omega_{\bar{\beta}}\|^2 + k(\omega^\#, \omega^\#) - k^*(\omega^\#, \omega^\#) + \frac{1}{2}t(\omega^\#, \omega^\#) \right] dV - \\
& - \int_{\mathbf{M}} \left\{ \frac{1}{2}(\delta\omega)^2 + \frac{1}{2}(\delta(J\omega))^2 - 2\operatorname{Re} \left[T_{\alpha\beta}^\sigma \omega_\sigma (D^\alpha \omega^\beta - D^\beta \omega^\alpha) \right] \right\} dV = 0. \\
(4.23) \quad & \int_{\mathbf{M}} \left\{ k(\omega^\#, \omega^\#) - k^*(\omega^\#, \omega^\#) + \frac{1}{2}2\operatorname{Re} \left[T_{\alpha\beta}^\sigma \omega_\sigma (D^\alpha \omega^\beta - D^\beta \omega^\alpha) \right] \right\} dV + \\
& + \int_{\mathbf{M}} 2\operatorname{Re} \left[T_{\alpha\beta}^\sigma D^\alpha \omega_\sigma \omega^\beta \right] dV = 0
\end{aligned}$$

Proof: The formula (4.22) follows immediately from (3.14) and (3.9).

To prove (4.23) we consider the following real 1-form

$$\psi = T_{\alpha\beta}^\sigma \omega_\sigma \omega^\beta dz^\alpha + T_{\bar{\alpha}\bar{\beta}}^{\bar{\sigma}} \omega_{\bar{\sigma}} \omega^{\bar{\beta}} d\bar{z}^{\bar{\alpha}}.$$

Applying (2.4) we have

$$\begin{aligned}
(4.24) \quad & -\delta\psi = 2\operatorname{Re} \left[D^\alpha T_{\alpha\beta}^\sigma \omega_\sigma \omega^\beta \right] + \\
& + 2\operatorname{Re} \left[T_{\alpha\beta}^\sigma D^\alpha \omega_\sigma \omega^\beta \right] + \operatorname{Re} \left[T_{\alpha\beta}^\sigma \omega_\sigma (D^\alpha \omega^\beta - D^\beta \omega^\alpha) \right]
\end{aligned}$$

From the second Bianchi identity we get

$$(4.25) \quad 2\operatorname{Re} \left[D^\alpha T_{\alpha\beta}^\sigma \omega_\sigma \omega^\beta \right] = s(\omega^\#, \omega^\#) - k^*(\omega^\#, \omega^\#).$$

Substituting (4.25) into (4.24) and integrating the obtained equality over \mathbf{M} we obtain (4.23) **Q.E.D.**

We define the tensor H by the equality

$$(4.26) \quad H(X, Y) := k(X, Y) - k^*(X, Y) - \frac{1}{2}t(X, Y), \quad X, Y \in \mathbf{X}\mathbf{M}$$

From this definition it follows that the tensor H is symmetric and J -invariant. We have

Theorem 4.2 *Let (\mathbf{M}, J, g) be a compact balanced Hermitian manifold.*

- i) A harmonic 1-form ω is analytic iff $\int_{\mathbf{M}} H(\omega^\#, \omega^\#) dV = 0$;*
- ii) An analytic 1-form ω is harmonic iff $\int_{\mathbf{M}} H(\omega^\#, \omega^\#) dV = 0$;*

Proof: The theorem follows from the following two lemmas:

Lemma 4.3 *Let ω be a harmonic 1-form on a compact balanced Hermitian manifold. Then we have*

$$(4.27) \quad \int_{\mathbf{M}} \left[\|D_\alpha \omega_{\bar{\beta}}\|^2 + H(\omega^\#, \omega^\#) \right] dV = 0.$$

Lemma 4.4 *Let ω be an analytic 1-form on a compact balanced Hermitian manifold. Then we have*

$$\int_{\mathbf{M}} \left[\frac{1}{2} \|D_\alpha \omega_\beta - D_\beta \omega_\alpha + T_{\alpha\beta}^\sigma \omega_\sigma\|^2 + H(\omega^\#, \omega^\#) \right] dV = 0.$$

The proof of Lemma 4.3 follows after substitution of (4.19), (4.20) and (4.21) into (4.22). Combining (4.23) with (4.22) and using (4.17) we get the proof of Lemma 4.4. This completes the proof of Theorem 4.2. **Q.E.D.**

Remark 4.5 *On a Kaehler manifold the tensor H vanishes identically and Theorem 4.2 implies the well known result that on a compact Kaehler manifold every harmonic 1-form is analytic and vice versa.*

Using Hodge theory (see e.g. [7]) from Theorem 4.2 we obtain

Corollary 4.6 *For a compact balanced Hermitian manifold with zero tensor H the de Rham cohomology group $H^1(\mathbf{M}, \mathbf{R})$ is isomorphic to the Dolbeault cohomology group $H^{1,0}(\mathbf{M}, \mathbf{C})$.*

Now we can state our main result (vanishing theorem of Bochner type)

Theorem 4.7 *Let (\mathbf{M}, J, g) be a compact balanced Hermitian manifold.*

i) If the tensor H is positive semi-definite then every analytic 1-form ω is harmonic and vice versa, every harmonic 1-form ω is analytic; moreover in the two cases $H(\omega^\#, \omega^\#) = 0$.

ii) If the tensor H is positive definite on \mathbf{M} then:

- *there are no harmonic 1-forms on \mathbf{M} ;*
- *there are no analytic 1-forms on \mathbf{M} .*

Proof: The proof of this theorem follows immediately from Lemma 4.3 and Lemma 4.4. **Q.E.D.**

Applying the Hodge theory we get

Theorem 4.8 *Let (\mathbf{M}, J, g) be a compact balanced Hermitian manifold with positive definite tensor H . Then*

- i) the first Betti number $b_1 = \dim H^1(\mathbf{M}, \mathbf{R}) = 0$;*
- ii) the Hodge number $h^{1,0} = \dim H^{1,0}(\mathbf{M}, \mathbf{C}) = 0$.*

Using (3.9) we can state Lemma 4.3 as

Lemma 4.9 *Let ω be a harmonic 1-form on a compact balanced Hermitian manifold. Then the following formula is true*

$$(4.28) \quad \int_{\mathbf{M}} \left[\|D_\alpha \omega_\beta\|^2 + k(\omega^\#, \omega^\#) - \frac{1}{2} t(\omega^\#, \omega^\#) \right] dV = 0$$

From (4.28) we get the following vanishing theorem of Bochner type

Theorem 4.10 *Let (\mathbf{M}, J, g) be a compact balanced Hermitian manifold.*

i) If the tensor $k - \frac{1}{2}t$ is positive semi-definite then the vector field $\omega^\#$ corresponding to any harmonic 1-form ω is holomorphic vector field and

$$(k - \frac{1}{2}t)(\omega^\#, \omega^\#) = 0.$$

ii) If the tensor $k - \frac{1}{2}t$ is positive definite on \mathbf{M} then there are no harmonic 1-forms on \mathbf{M} and consequently $b_1 = 0$.

Remark 4.11 *If the tensor $k - \frac{1}{2}t$ is positive definite on \mathbf{M} then the first Chern form κ is positive, by the properties of t . From the Calabi-Yau-Aubin theory (see[7]) there exists a Kaehler metric on \mathbf{M} with positive Ricci tensor and the last conclusion in ii) of Theorem 4.10 follows from the classical vanishing theorem of Bochner.*

It is well known that on a compact Hermitian manifold if the mean curvature k^* is positive semi-definite and it is positive definite in one point then there are no holomorphic $(p, 0)$ -forms ([14, 8], see also [17, 13]), in particular there are no holomorphic $(1, 0)$ -forms (the latter fact for balanced manifolds follows also from (3.9)). In view of Theorem 4.7, we can generalize the latter fact for compact balanced manifolds as follows

Theorem 4.12 *If on a compact balanced Hermitian manifold the mean curvature satisfies the condition*

$$k^*(X, X) < k(X, X) - \frac{1}{2}t(X, X), \quad X \in \mathbf{X}\mathbf{M}$$

then there are neither holomorphic $(1, 0)$ -forms nor harmonic 1-forms on \mathbf{M} .

As a corollary we also have

Theorem 4.13 *Let (\mathbf{M}, J, g) be a compact balanced Hermitian manifold. If the tensors H and k^* are positive semi-definite on \mathbf{M} and k^* is positive definite in one point then there are no harmonic 1-forms on \mathbf{M} and consequently $b_1 = 0$*

References

- [1] L.Alessandrini and G.Bassanelli, *Positive $\partial\bar{\partial}$ -closed currents and non-Kaehler geometry*, to appear in The J. of Geom. Analysis.
- [2] L.Alessandrini and G.Bassanelli, *Smooth proper modification of compact Kaehler manifolds*, Proc. of the Intern. Workshop on Complex Analysis, Wupertal 1990, Complex Analysis, Aspects of Math. E vol. **17**, Germany 1991.

- [3] L.Alessandrini and G.Bassanelli, *Metric properties of manifolds bimeromorphic to compact Kaehler manifolds*, J. Diff. Geometry **37**(1993), 95 – 121.
- [4] B.Alexandrov, G.Grantcharov, S.Ivanov, *Curvature properties of twistor spaces of quaternionic Kaehler manifolds*, to appear.
- [5] H.Akbahr-Zadeh, *Transformations holomorphiquement projectives des varietes hermitiennes et kaehleriennes*, J. Math. pures at appl. **67**(1988), 237 – 261.
- [6] A.Balas , *Compact Hermitian manifolds of constant holomorphic sectional curvature.*, Math. Z. **189**(1985), 193 – 210.
- [7] A.Besse, *Einstein manifolds*, Springer, Berlin 1987.
- [8] P.Gauduchon, *Fibres hermitiennes a endomorphisme de Ricci non-negatif.*, Bull. Soc. Math. France **105** (1977), 113 – 140.
- [9] P.Gauduchon, *Structures de Weyl et theoremes d’anulation sur une variete conforme autoduale*, Ann. Scuola Norm. Sup. Pisa, Serie IV, vol. XVIII, Fasc. **4**(1991), 563 – 629.
- [10] S.Goldberg, *Curvature and homology*, New- York: Academic Press 1962.
- [11] F.Hirzebruch, *New geometric methods in algebraic geometry*, Springer-Verlag, Berlin, Heidelberg, New-York (1966).
- [12] S.Kobayashi, *Transformation groups in Differential Geometry*, Springer-Verlag, Berlin, Heidelberg, New-York (1972).
- [13] S.Kobayashi, *Differential Geometry of complex vector bundles*, Iwanami Shoten, Publishers and Princeton Univ. Press. 1987.
- [14] S.Kobayashy and H.Wu, *On holomorphic sections of certain Hermitian vector bundles*, Math. Ann. **189** (1970), 1 – 4.
- [15] M.L.Michelson, *On the existence of special metrics in complex geometry*, Acta Math. **143**(1983), 261 – 295.
- [16] M.Pontecorvo, *Complex structures on quaternionic manifold*, Diff. Geom. and its Appl. **4**(1994), 163 – 177.
- [17] H.Wu, *Bochner technique in Differential Geometry*, Math. Reports, vol.3, part 2, 1988.
- [18] K.Yano, *The theory of Lie derivatives and its applications*, Amsterdam, North-Holland 1957.

- [19] K.Yano, *Differential geometry on Complex and Almost Complex spaces*, Pergamon Press, 1965.

Author's address:

Georgi Ganchev,
Bulgarian Academy of Science, Institute of Mathematics, Acad. G.Bonchev Str.,blok
8, 1113 Sofia BULGARIA

Stefan Ivanov
University of Sofia, Faculty of Mathematics and Informatics, Department of Geom-
etry, bul. James Bouchier 5, 1126 Sofia, BULGARIA.
E-mail: ivanovsp@fmi.uni-sofia.bg